College Name - S.S. College J. Bad

Deptt - Maths - Time - 12/5/12

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Whats app No —

B.Sc Part I

Topic - Symmetric Functions of the Roots (theory of equation)

Problem :→

Study Material - Key Facts

1. If a function remain unchanged in value when any two of its arguments are interchanged, is called Symmetric Function.

2. In a Symmetric Function the sum of the Exponents of each term is always Constant.

3. The sum of the Exponents of a Symmetric function is called the weight of the function.

4. The highest Power of an arguments of a Symmetric function is called its order.

5. Ex :→ 1. $\Sigma\alpha^3\beta^2\gamma$ is a Symmetric Function which is of 3rd order and weight = 6

2. $\Sigma\alpha^2\beta\gamma$ order = 2, weight = 4

6. If $x^3 + px^2 + qx + r = 0 \Rightarrow \Sigma\alpha = -p$

$\Sigma\alpha\beta = q$

$\alpha\beta\gamma = -r$

it may be written as

$x^3 - \Sigma\alpha\, x^2 + \Sigma\alpha\beta\, x - \alpha\beta\gamma = 0$

6. If $x^4 + px^3 + qx^2 + rx + s = 0$ be a Biquadratic Equation

then $\Sigma\alpha = -p$ $\quad \Sigma\alpha\beta\gamma = -r$

$\Sigma\alpha\beta = q$ $\quad \alpha\beta\gamma\delta = s$

it may be written as

$$x^4 - \Sigma\alpha\, x^3 + \Sigma\alpha\beta\, x^2 - \Sigma\alpha\beta\gamma\, x + \alpha\beta\gamma\delta = 0$$

7. for Cubic Equation

(i) $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

(ii) $\Sigma\alpha^2\beta = \Sigma\alpha\,\Sigma\alpha\beta - 3\alpha\beta\gamma$

(iii) $\Sigma\alpha^3 = \Sigma\alpha\,\Sigma\alpha^2 - \Sigma\alpha^2\beta$

(iv) $\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma\,\Sigma\alpha$

(v) $\Sigma\frac{1}{\alpha} = \frac{\Sigma\alpha\beta}{\alpha\beta\gamma}$

(vi) $\Sigma\frac{1}{\alpha\beta} = \frac{\Sigma\alpha}{\alpha\beta\gamma}$

(vii) $\Sigma\frac{1}{\alpha^2} = \left(\Sigma\frac{1}{\alpha}\right)^2 - 2\,\Sigma\frac{1}{\alpha\beta}$

for Biquadratic Equation

(i) $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

(ii) $\Sigma\alpha^2\beta = \Sigma\alpha\,\Sigma\alpha\beta - 3\Sigma\alpha\beta\gamma$

(iii) $\Sigma\alpha^3 = \Sigma\alpha\,\Sigma\alpha^2 - \Sigma\alpha^2\beta$

(iv) $\Sigma\alpha^2\beta\gamma = \Sigma\alpha\,\Sigma\alpha\beta\gamma - 4\alpha\beta\gamma\delta$

(v) $\Sigma\alpha^3\beta = \Sigma\alpha^2\,\Sigma\alpha\beta - \Sigma\alpha^2\beta\gamma$

(vi) $\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\Sigma\alpha^2\beta\gamma - 6\alpha\beta\gamma\delta$

(v) $\Sigma\alpha^4 = (\Sigma\alpha^2)^2 - 2\Sigma\alpha^2\beta^2$

(vi) $\Sigma\alpha^2\beta^2\gamma^2 = (\Sigma\alpha\beta\gamma)^2 - 2\alpha\beta\gamma\delta\,\Sigma\alpha\beta$

(vii) $\Sigma\frac{1}{\alpha} = \frac{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma\delta}$

(viii) $\Sigma\frac{1}{\alpha\beta} = \frac{\Sigma\alpha\beta}{\alpha\beta\gamma\delta}$

(IX) $\sum \frac{1}{\alpha^2} = \left(\sum \frac{1}{\alpha}\right)^2 - 2 \sum \frac{1}{\alpha\beta}$

(X) $\sum \frac{\alpha}{\beta} = \sum \alpha \sum \frac{1}{\alpha} - 4$

(XI) $\sum \frac{\alpha^2}{\beta^2} = \sum \alpha^2 \sum \frac{1}{\alpha^2} - 4$

(XI) $\sum \frac{\alpha\beta}{\gamma^2} = \sum \alpha\beta \sum \frac{1}{\alpha^2} - \sum \alpha/\beta$

Problem 1. → If α, β, γ be the roots of the Cubic $x^3 + px^2 + qx + r = 0$

Express the Symmetric Functions

(I) $\sum \alpha^2$ (II) $\sum \alpha^2\beta$ Imp

(III) $\sum \alpha^3$ (IV) $\sum \frac{1}{\alpha}$ (V) $\sum \frac{1}{\alpha\beta}$

(VI) $\sum \frac{1}{\alpha^2}$ (VII) $\sum \alpha^2\beta^2$ (VIII) $\sum \frac{\alpha^2+\beta^2}{\alpha\beta}$ Imp.

Here Given Cubic Equation is

$x^3 + px^2 + qx + r = 0$

Here by the question α, β, γ are its roots

$\therefore \sum \alpha = -p \quad \sum \alpha\beta = q \quad \alpha\beta\gamma = -r$

Now (I) $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$

$= p^2 - 2q$

Since $\sum \alpha^2 = \alpha^2 + \beta^2 + \gamma^2$

as we know $(a+b+c)^2 - 2(ab+bc+ca) = a^2+b^2+c^2$

$(\sum \alpha)^2 - 2\sum \alpha\beta = \sum \alpha^2$

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(ii) $\sum \alpha^2\beta = \sum\alpha \sum\alpha\beta - 3\alpha\beta\gamma$

$= (-p)(q) - 3(r)$

$= -pq + 3r$

$= 3r - pq$

(iii) $\sum\alpha^3 = \sum\alpha \sum\alpha^2 - 3\sum\alpha^2\beta$

$= (-p)(p^2-2q) - 3\cdot(3r-pq)$

$= -p^3 + 2pq - 3r + pq$

$= -p^3 + 2pq - 3r + pq$

$= 3pq - p^3 - 3r$

(iv) $\sum\frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$

$= \sum\alpha\beta \cdot \frac{1}{\alpha\beta\gamma}$

$= q \cdot \frac{1}{-r}$

$= -\frac{q}{r}$

(v) $\sum\frac{1}{\alpha\beta} = \frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$

$= \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma}$

$= \frac{\sum\alpha}{\alpha\beta\gamma} = \frac{(-p)}{-r} = \frac{p}{r}.$

(VI) $\Sigma \frac{1}{\alpha^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

$= \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)^2 - 2\left(\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}\right)$

$= \left(\Sigma \frac{1}{\alpha}\right)^2 - 2 \frac{\Sigma \alpha}{\alpha\beta\gamma}$

$= \left(-\frac{q}{r}\right)^2 - 2 \frac{(-p)}{(-r)}$

$= \frac{q^2}{r^2} - 2\frac{p}{r}$

$= \frac{q^2 - 2pr}{r^2}$

(VII) $\Sigma \alpha^2\beta^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta\cdot\beta\gamma + \beta\gamma\cdot\gamma\alpha + \gamma\alpha\,\alpha\beta)$

$= (\Sigma\alpha\beta)^2 - 2[\alpha\beta^2\gamma + \beta\gamma^2\alpha + \gamma\alpha^2\beta]$

$= (\Sigma\alpha\beta)^2 \cancel{- 2\Sigma\alpha^2\beta\gamma} - 2\alpha\beta\gamma(\beta+\gamma+\alpha)$

$= (\Sigma\alpha\beta)^2 - 2\Sigma\alpha \cdot \alpha\beta\gamma$

$= (q)^2 - 2(-p)(-r)$

$= q^2 - 2pr$

(VIII) $\Sigma \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{\alpha^2+\beta^2}{\alpha\beta} + \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\gamma^2+\alpha^2}{\gamma\alpha}$

$= \frac{\gamma(\alpha^2+\beta^2) + \alpha(\beta^2+\gamma^2) + \beta(\gamma^2+\alpha^2)}{\alpha\beta\gamma}$

$= \frac{\Sigma\alpha^2\beta}{\alpha\beta\gamma} = \frac{3r - pq}{-r} = \frac{pq - 3r}{r}$

Problem 2. for the Cubic $x^3+px^2+qx+r=0$ find the value of the following symmetric functions :→

(i) $\sum \frac{\alpha^2+\beta\gamma}{\beta+\gamma}$ (ii) $\sum(\alpha+\beta-\gamma)^3$

(iii) $\sum\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$ (iv) $\sum \frac{\beta^2+\alpha^2}{\beta+\alpha}$

Since the Cubic Equation $x^3+px^2+qx+r=0$

$\sum\alpha = -p \qquad \sum\alpha\beta = q \qquad \alpha\beta\gamma = -r$

(i) $\sum \frac{\alpha^2+\beta\gamma}{\beta+\gamma} = \frac{\alpha^2+\beta\gamma}{\beta+\gamma} + \frac{\beta^2+\gamma\alpha}{\gamma+\alpha} + \frac{\gamma^2+\alpha\beta}{\alpha+\beta}$

$= \frac{(\alpha^2+\beta\gamma)(\gamma+\alpha)(\alpha+\beta) + (\beta^2+\gamma\alpha)(\beta+\gamma)(\alpha+\beta) + (\gamma^2+\alpha\beta)(\beta+\gamma)(\gamma+\alpha)}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}$

$= \frac{\sum(\alpha^2+\beta\gamma)(\gamma+\alpha)(\alpha+\beta)}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}$ ——— (1)

$N^r = \cancel{\sum\alpha^2\beta} = \sum(\alpha^2+\beta\gamma)(\gamma+\alpha)(\alpha+\beta)$

$= \sum(\alpha^2+\beta\gamma)(\alpha^2+\alpha\beta+\beta\gamma+\gamma\alpha)$

$= \sum(\alpha^2+\beta\gamma)(\alpha^2+\sum\alpha\beta)$

$= \sum(\alpha^2+\beta\gamma)(\alpha^2+q)$

$= \sum\alpha^4 + q\sum\alpha^2 + \sum\alpha^2\beta\gamma + q\sum\beta\gamma.$

Now $\sum\alpha^4 = \alpha^4+\beta^4+\gamma^4$

$= (\alpha^2+\beta^2+\gamma^2)^2 - 2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)$

$= (\sum\alpha^2)^2 - 2\sum\alpha^2\beta^2 = (p^2-2q)^2 - 2(q^2-pr)$

Now $\sum\alpha^2\beta^2 = (\sum\alpha\beta)^2 - 2\sum\alpha^2\beta\gamma.$

$= (\sum\alpha\beta)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$= q^2 - 2(-r)(-p) = q^2-2pr$

$= \cancel{2pr+q^2} \quad \cancel{q^2 2p}$

~~Hence~~ $\cancel{N^r = (p^2-2q)^2 - 2(q^2-2pr) + q^2}$

$N^r = (p^2-2q)^2 - 2(q^2-2pr) + q(p^2-2q) + (-r)(-p) + q^2$

$= p^4 - 4p^2q + 4q^2 - 2q^2 + 4pr + p^2q - 2q^2 + pr + q^2$

$= p^4 - 3p^2q + 5pr + q^2$

$$Dr = (\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$$

$$= (\alpha+\beta+\gamma-\gamma)(\alpha+\beta+\gamma-\alpha)(\alpha+\beta+\gamma-\beta)$$

$$= (\Sigma\alpha-\gamma)(\Sigma\alpha-\alpha)(\Sigma\alpha-\beta)$$

$$= (-p-\gamma)(-p-\alpha)(-p-\beta)$$

$$= -\left[(p+\alpha)(p+\beta)(p+\gamma)\right]$$

$$= -\left[p^3 + p^2\Sigma\alpha + p\Sigma\alpha\beta + \alpha\beta\gamma\right]$$

$$= -\left[\cancel{p^3} - \cancel{p^3} + pq - r\right] = r - pq$$

Hence $$\Sigma \frac{\alpha^2+\beta\gamma}{\beta+\gamma} = \frac{p^4 - 3p^2q + 5pr + q^2}{r-pq}.$$